

# KKR TYPE BIJECTION FOR THE EXCEPTIONAL AFFINE ALGEBRA $E_6^{(1)}$

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**ABSTRACT.** For the exceptional affine type  $E_6^{(1)}$  we establish a statistic-preserving bijection between the highest weight paths consisting of the simplest Kirillov-Reshetikhin crystal and the rigged configurations. The algorithm only uses the structure of the crystal graph, hence could also be applied to other exceptional types.

## 1. INTRODUCTION

In a pioneering work [15] Kerov, Kirillov and Reshetikhin introduced a new combinatorial object, called rigged configuration, through Bethe ansatz analysis of the Heisenberg spin chain, and constructed a bijection between rigged configurations and semistandard tableaux. One of the amazing properties of the rigged configuration is that it possesses a natural statistic and the statistic coincides with the charge by Lascoux and Schützenberger [21] on the tableau under the bijection. Subsequently, Nakayashiki and Yamada [25] studied the meaning of the charge in terms of Kashiwara's crystal bases. They considered the crystal base  $B_l$  of the  $l$ -fold symmetric tensor representation of the  $n$ -dimensional irreducible  $U_q(\widehat{\mathfrak{sl}}_n)$ -module. For the tensor product  $B_l \otimes B_{l'}$  an integer-valued function  $H$ , called energy function, is defined via the  $q \rightarrow 0$  limit of the quantum  $R$ -matrix. Using this  $H$  they constructed a function  $D$  on the multiple tensor product  $B_{l_1} \otimes \cdots \otimes B_{l_m}$ . They then showed that under a certain bijection sending highest weight vectors or paths of  $B_{l_1} \otimes \cdots \otimes B_{l_m}$  to semistandard tableaux, the value of  $D$  agrees with the charge, thereby proving that the well-known Kostka polynomial is represented as a generating function of highest weight paths with statistic  $D$ . This generating function is denoted by  $X$  and the one of rigged configurations by  $M$ . The equality  $X = M$  was extended to the most general case for affine type  $A$  in [17]. See also [30] for review.

It did not take long before this kind of equality was conjectured to exist for other affine types. For the  $X$  side, crystal bases for some finite-dimensional modules, which are now called Kirillov-Reshetikhin (KR) modules, for quantum affine algebras have been discovered in [13]. For the  $M$  side, the existence of KR modules were conjectured and a formula to count the number of rigged configurations were presented in [16]. Introducing an appropriate  $q$ -analogue for the formula, the  $X = M$  conjecture [8, 7] was presented. Imitating the one by KKR a bijection between rigged configurations and highest weight paths consisting of elements of KR crystals for other nonexceptional affine types was subsequently constructed in [28, 29, 31]. We note that these bijections have an important application for

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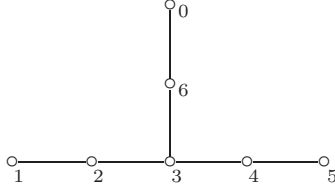


FIGURE 1. Dynkin diagram for  $E_6^{(1)}$

the analysis of the ultra-discrete integrable systems, also called box-ball systems [4, 6, 9]. In such systems rigged configurations give the complete set of the action and angle variables [19, 20].

In this paper we consider the exceptional affine algebra of type  $E_6^{(1)}$ . The KR crystal we deal with is the simplest one denoted in our notation by  $B^{1,1}$ , whose crystal structure was revealed in [22, 5]. We construct a map  $\Phi$  from rigged configurations to highest weight elements of  $(B^{1,1})^{\otimes L}$  by executing a fundamental procedure  $\delta$  repeatedly. We then show  $\Phi$  is a statistic-preserving bijection (Theorem 3.2). It is worth mentioning that our procedure only uses the crystal graph structure of the KR crystal  $B^{1,1}$ , hence similar constructions could be possible for other exceptional types.

We remark that recently Naoi [26] solved, with the help of the results in [3] and [23], the  $X = M$  conjecture for all untwisted affine types when the tensor product of KR crystals is of the form  $B^{r_1,1} \otimes \cdots \otimes B^{r_l,1}$  by showing both  $X$  and  $M$  are equal to the graded character of a Weyl module, a finite-dimensional current algebra representation defined in [2]. Hence his result includes ours as a special case. However, we think our direct method is also important, since it could also be used for more general cases by cutting larger KR crystals as in [17].

## 2. QUANTUM AFFINE ALGEBRA AND CRYSTAL

**2.1. Affine algebra  $E_6^{(1)}$ .** We consider in this paper the exceptional affine algebra  $E_6^{(1)}$ . The Dynkin diagram is depicted in Figure 1. Note that we follow [11] for the labeling of the Dynkin nodes. It is different from that in [1] or [5]. Let  $I$  be the index set of the Dynkin nodes, and let  $\alpha_i, \alpha_i^\vee, \Lambda_i$  ( $i \in I$ ) be simple roots, simple coroots, fundamental weights, respectively. Following the notation in [11] we denote the projection of  $\Lambda_i$  onto the weight space of  $E_6$  by  $\overline{\Lambda}_i$  ( $i \in I_0$ ) and set  $\overline{P} = \bigoplus_{i \in I_0} \mathbb{Z} \overline{\Lambda}_i$ ,  $\overline{P}^+ = \bigoplus_{i \in I_0} \mathbb{Z}_{\geq 0} \overline{\Lambda}_i$ . Let  $(C_{ij})_{i,j \in I}$  stand for the Cartan matrix for  $E_6^{(1)}$ . For  $i, j \in I$ ,  $i \sim j$  means  $C_{ij} = -1$ , namely, the nodes  $i$  and  $j$  are adjacent in the Dynkin diagram of  $E_6^{(1)}$ .

**2.2. KR crystal.** Let  $\mathfrak{g}$  be any affine algebra and  $U'_q(\mathfrak{g})$  the corresponding quantized enveloping algebra without the degree operator. Among finite-dimensional  $U'_q(\mathfrak{g})$ -modules there is a distinguished family called Kirillov-Reshetikhin (KR) modules [18, 24, 10]. One of the remarkable properties of KR modules is the existence of a crystal basis [14] called a KR crystal. It was conjectured in [8, 7], and recently settled for all nonexceptional types in [27]. The KR crystal is indexed by  $(a, i)$  ( $a \in I_0, i \in \mathbb{Z}_{>0}$ ) and denoted by  $B^{a,i}$ . For exceptional types the KR crystal is

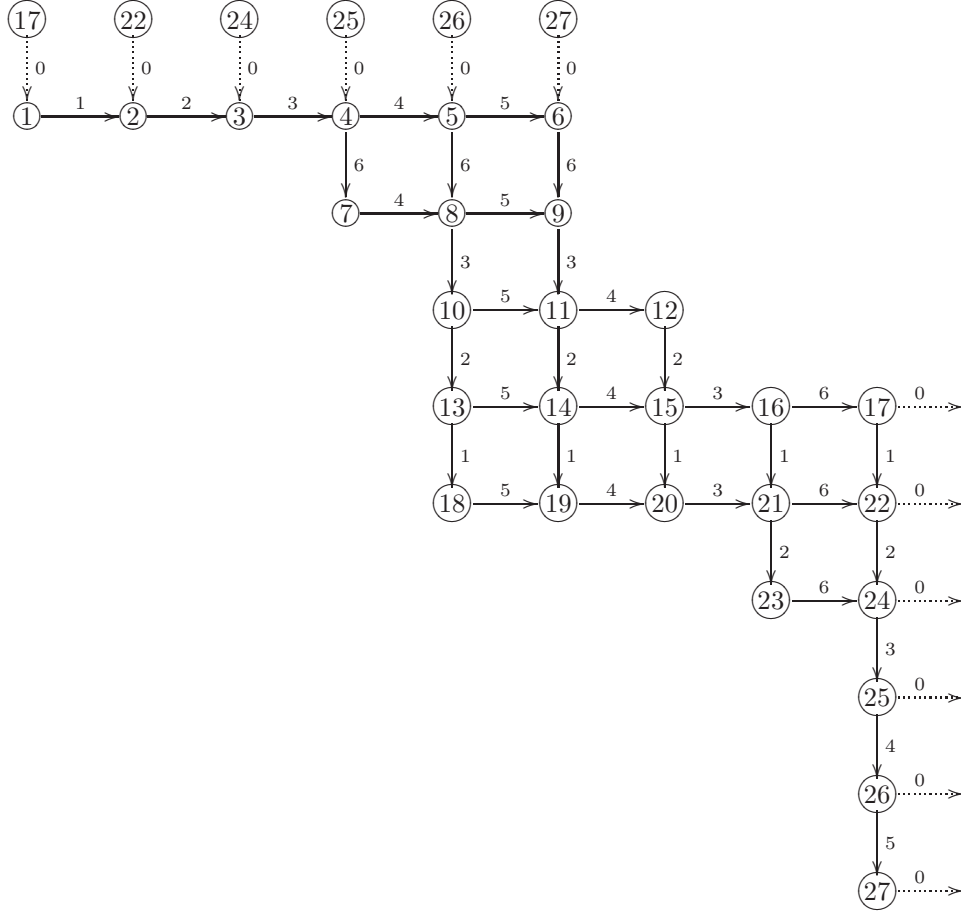


FIGURE 2. Crystal graph for  $B^{1,1}$

known to exist when the KR module is irreducible or the index  $a$  is adjacent to 0 [13]. Recently, the explicit crystal structure of all such cases of type  $E_6^{(1)}$  was clarified in [5].

The KR crystal we are interested in in this paper is an  $E_6^{(1)}$ -crystal  $B^{1,1}$ , whose crystal structure was clarified in [5]. The crystal structure of  $B^{1,1}$  is depicted in Figure 2. Here vertices in the graph signify elements of  $B^{1,1}$  and  $b \xrightarrow{i} b'$  stands for  $f_i b = b'$  or equivalently  $b = e_i b'$ . We adopt the original convention for the tensor product of crystals. Namely, if  $B_1$  and  $B_2$  are crystals, then for  $b_1 \otimes b_2 \in B_1 \otimes B_2$  the action of  $e_i$  is defined as

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2 & \text{else,} \end{cases}$$

where  $\varepsilon_i(b) = \max\{k \mid e_i^k b \neq 0\}$  and  $\varphi_i(b) = \max\{k \mid f_i^k b \neq 0\}$ .

By glancing at Figure 2, one obtains the following lemma which will be used to prove our main theorem. Let  $B_0$  be the subgraph obtained by ignoring the 0-arrows

from  $B$ . A route is a sequence  $(\beta_1, \dots, \beta_l)$  of arrows such that the sink of  $\beta_j$  is the source of  $\beta_{j+1}$  for  $j = 1, \dots, l-1$ .

**Lemma 2.1.** *The graph  $B_0$  has the following features.*

- (1) *Suppose the initial arrow of a route  $R$  has the same color  $a$  as the terminal arrow and there is no intermediate arrow of color  $a$ . Then there are exactly two arrows  $\beta_i$  ( $i = 1, 2$ ) of color  $b_i$  such that  $b_i \sim a$  in  $R$ .*
- (2) *Let  $R$  be a route starting from  $\Phi$ ,  $(a_1, \dots, a_l)$  the colors from the initial arrow to the terminal one in  $R$ . Then we have*

$$\sum_{j=1}^{l-1} C_{a_j a_l} = \delta_{a_l, 1} - 1.$$

- (3) *Let  $R$  be a route of two steps with colors  $(a, b)$  such that  $b \not\sim a$ . Then there exists a route  $R'$  with colors  $(b, a)$  starting and terminating at the same vertices as  $R$ .*
- (4) *Let  $R$  be a route of colors  $(a_1, \dots, a_l)$ . Let  $v_i$  be the source of the arrow of color  $a_i$  ( $i = 1, \dots, l$ ). Suppose  $a_1 \sim a_l$  and  $a_i \not\sim a_l$  for any  $i = 2, \dots, l-1$ . Then there is an arrow of color  $a_l$  starting from  $v_i$  for any  $i = 2, \dots, l-1$ .*

*Proof.* (1) and (3) can be checked by direct observations. (2) and (4) are derived from (1) and (3).  $\square$

In what follows in this paper we assume  $B = B^{1,1}$ . The set of classically restricted paths in  $B^{\otimes L}$  of weight  $\lambda \in \overline{P}^+$  is by definition

$$(2.1) \quad \mathcal{P}(\lambda, L) = \{b \in B^{\otimes L} \mid \text{wt}(b) = \lambda \text{ and } e_i b = 0 \text{ for all } i \in I_0\}.$$

One may check that the following are equivalent for  $b = b_1 \otimes b_2 \otimes \dots \otimes b_L \in B^{\otimes L}$  and  $\lambda \in \overline{P}^+$ .

- (1)  $b$  is a classically restricted path of weight  $\lambda \in \overline{P}^+$ .
- (2)  $b_1 \otimes \dots \otimes b_{L-1}$  is a classically restricted path of weight  $\lambda - \text{wt}(b_L)$ , and  $\varepsilon_i(b_L) \leq \langle \lambda - \text{wt}(b_L), \alpha_i^\vee \rangle$  for all  $i \in I_0$ .

The weight function  $\text{wt} : B \rightarrow \overline{P}$  is given by  $\text{wt}(b) = \sum_{i \in I} (\varphi_i(b) - \varepsilon_i(b)) \Lambda_i$ . The weight function  $\text{wt} : B^{\otimes L} \rightarrow \overline{P}$  is defined by  $\text{wt}(b_1 \otimes \dots \otimes b_L) = \sum_{j=1}^L \text{wt}(b_j)$ .

**Example 2.2.** The element

$$b = \textcircled{1} \cdot \textcircled{2} \cdot \textcircled{3} \cdot \textcircled{16} \cdot \textcircled{2} \cdot \textcircled{24}$$

of  $B^{\otimes 6}$  is a classically restricted path of weight  $\overline{\Lambda}_3$ . The dot  $\cdot$  signifies  $\otimes$ .

**2.3. One-dimensional sums.** The energy function  $D : B^{\otimes L} \rightarrow \mathbb{Z}$  gives the grading on  $B^{\otimes L}$ . In our case where a path is an element of the tensor product of a single KR crystal it takes a simple form. Due to the existence of the universal  $R$ -matrix and the fact that  $B \otimes B$  is connected, by [12] there is a unique (up to global additive constant) function  $H : B \otimes B \rightarrow \mathbb{Z}$  called the local energy function, such that

$$(2.2) \quad H(e_i(b \otimes b')) = H(b \otimes b') + \begin{cases} 1 & \text{if } i = 0 \text{ and } e_0(b \otimes b') = e_0 b \otimes b' \\ -1 & \text{if } i = 0 \text{ and } e_0(b \otimes b') = b \otimes e_0 b' \\ 0 & \text{otherwise.} \end{cases}$$

We normalize  $H$  by the condition

$$(2.3) \quad H(\textcircled{1} \otimes \textcircled{1}) = 0.$$

More specifically, the value of  $H$  is calculated as follows. Firstly, one knows the crystal graph of  $B_0 \otimes B_0$  decomposes into three connected components as

$$B_0 \otimes B_0 = B(2\overline{\Lambda}_1) \oplus B(\overline{\Lambda}_1 + \overline{\Lambda}_2) \oplus B(\overline{\Lambda}_1 + \overline{\Lambda}_5),$$

where  $B(\lambda)$  stands for the highest weight  $E_6$ -crystal of highest weight  $\lambda$  and the highest weight vector of each component is given by  $\Phi \otimes \Phi, \Phi \otimes \mathfrak{Q}, \Phi \otimes \mathfrak{U}$ .  $H$  is constant on each component, and takes the value  $0, -1, -2$ , respectively. One can confirm it from the fact that  $e_0(\Phi \otimes \Phi) = \Phi \otimes \mathfrak{U}$  and  $e_0(\Phi \otimes \mathfrak{Q}) = \Phi \otimes \mathfrak{U}$  belong to the second and third component.

With this  $H$  the energy function  $D$  is defined by

$$(2.4) \quad D(b_1 \otimes \cdots \otimes b_L) = \sum_{j=1}^{L-1} (L-j) H(b_j \otimes b_{j+1}).$$

Define the one-dimensional sum  $X(\lambda, L; q) \in \mathbb{Z}_{\geq 0}[q^{-1}]$  by

$$(2.5) \quad X(\lambda, L; q) = \sum_{b \in \mathcal{P}(\lambda, L)} q^{D(b)}.$$

### 3. RIGGED CONFIGURATION AND THE BIJECTION

**3.1. The fermionic formula.** This subsection reviews the definition of the fermionic formula from [7, 8]. We at first provide the definition that is valid for any simply-laced affine type  $\mathfrak{g}$  and datum  $\mathbf{L}$ , and then restrict  $\mathfrak{g}$  and  $\mathbf{L}$  to  $E_6^{(1)}$  and the case corresponding to paths we consider in this paper. Fix  $\lambda \in \overline{P}^+$  and a matrix  $\mathbf{L} = (L_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$  of nonnegative integers, almost all zero. Let  $\nu = (m_i^{(a)})$  be another such matrix. Say that  $\nu$  is an admissible configuration if it satisfies

$$(3.1) \quad \sum_{\substack{a \in I_0 \\ i \in \mathbb{Z}_{>0}}} i m_i^{(a)} \alpha_a = \sum_{\substack{a \in I_0 \\ i \in \mathbb{Z}_{>0}}} i L_i^{(a)} \overline{\Lambda}_a - \lambda$$

and

$$(3.2) \quad p_i^{(a)} \geq 0 \quad \text{for all } a \in I_0 \text{ and } i \in \mathbb{Z}_{>0},$$

where

$$(3.3) \quad p_i^{(a)} = \sum_{j \in \mathbb{Z}_{>0}} \left( L_j^{(a)} \min(i, j) - \sum_{b \in I_0} (\alpha_a | \alpha_b) \min(i, j) m_j^{(b)} \right).$$

Write  $C(\lambda, \mathbf{L})$  for the set of admissible configurations for  $\lambda \in \overline{P}^+$  and  $\mathbf{L}$ . Define the charge of a configuration  $\nu$  by

$$(3.4) \quad c(\nu) = \frac{1}{2} \sum_{a, b \in I_0} \sum_{j, k \in \mathbb{Z}_{>0}} (\alpha_a | \alpha_b) \min(j, k) m_j^{(a)} m_k^{(b)} - \sum_{a \in I_0} \sum_{j, k \in \mathbb{Z}_{>0}} \min(j, k) L_j^{(a)} m_k^{(a)}.$$

Using (3.3)  $c(\nu)$  is rewritten as

$$(3.5) \quad c(\nu) = -\frac{1}{2} \left( \sum_{a \in I_0, i \in \mathbb{Z}_{>0}} p_i^{(a)} m_i^{(a)} + \sum_{a \in I_0, j, k \in \mathbb{Z}_{>0}} \min(j, k) L_j^{(a)} m_k^{(a)} \right).$$

The fermionic formula is then defined by

$$(3.6) \quad M(\lambda, \mathbf{L}; q) = \sum_{\nu \in C(\lambda, \mathbf{L})} q^{c(\nu)} \prod_{a \in I_0} \prod_{i \in \mathbb{Z}_{>0}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}.$$

We now set  $\mathfrak{g} = E_6^{(1)}$  and

$$(3.7) \quad L_i^{(a)} = L\delta_{a1}\delta_{i1} \quad (a \in I_0, i \in \mathbb{Z}_{>0}).$$

The latter restriction corresponds to considering paths in  $(B^{1,1})^{\otimes L}$ . By abuse of notation we denote the fermionic formula under the restriction (3.7) by  $M(\lambda, L; q)$ . Then the  $X = M$  conjecture of [8, 7] states in this particular case that

$$(3.8) \quad X(\lambda, L; q) = M(\lambda, L; q).$$

**3.2. Rigged configuration.** The fermionic formula  $M(\lambda, \mathbf{L}; q)$  can be interpreted using combinatorial objects called rigged configurations. These objects are a direct combinatorialization of the fermionic formula  $M(\lambda, \mathbf{L}; q)$ . Our goal is to prove (3.8) by defining a statistic-preserving bijection from rigged configurations to classically restricted paths. Let  $\nu = (m_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$  be an admissible configuration. We identify  $\nu$  with a sequence of partitions  $\{\nu^{(a)}\}_{a \in I_0}$  such that  $\nu^{(a)} = (1^{m_1^{(a)}} 2^{m_2^{(a)}} \dots)$ . Let  $J = \{J^{(a,i)}\}_{(a,i) \in I_0 \times \mathbb{Z}_{>0}}$  be a double sequence of partitions. Then a rigged configuration is a pair  $(\nu, J)$  subject to the restriction (3.1) and the requirement that  $J^{(a,i)}$  be a partition contained in a  $m_i^{(a)} \times p_i^{(a)}$  rectangle.

For a partition  $\mu$  and  $i \in \mathbb{Z}_{>0}$ , define

$$(3.9) \quad Q_i(\mu) = \sum_j \min(\mu_j, i),$$

the area of  $\mu$  in the first  $i$  columns. Then setting  $Q_i^{(a)} = Q_i(\nu^{(a)})$  the vacancy number (3.3) under the restriction (3.7) is rewritten as

$$(3.10) \quad p_i^{(a)} = L\delta_{a1} - 2Q_i^{(a)} + \sum_{b \sim a} Q_i^{(b)},$$

where  $b \sim a$  stands for  $C_{ba} = -1$  as defined in §2.1.

The set of rigged configurations for fixed  $\lambda$  and  $\mathbf{L}$  is denoted by  $\text{RC}(\lambda, \mathbf{L})$ . Then (3.6) is equivalent to

$$M(\lambda, \mathbf{L}; q) = \sum_{(\nu, J) \in \text{RC}(\lambda, \mathbf{L})} q^{c(\nu, J)}$$

where

$$(3.11) \quad c(\nu, J) = c(\nu) + |J|$$

with  $c(\nu)$  as in (3.4) and  $|J| = \sum_{(a,i) \in I_0 \times \mathbb{Z}_{>0}} |J^{(a,i)}|$ . The set  $\text{RC}(\lambda, \mathbf{L})$  with the restriction (3.7) is denoted by  $\text{RC}(\lambda, L)$ .

**Example 3.1.** A rigged configuration in  $\text{RC}(\bar{\Lambda}_3, 6)$  is illustrated below.

$$\begin{array}{ccccc} \begin{array}{c} 0 \\ 1 \end{array} \begin{array}{|c|c|} \hline & \\ \hline \end{array} 0 & \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{|c|c|} \hline & \\ \hline \end{array} 0 & \begin{array}{c} 1 \\ 0 \end{array} \begin{array}{|c|c|} \hline & \\ \hline \end{array} 1 & \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{|c|c|} \hline & \\ \hline \end{array} 0 & \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{|c|c|} \hline & \\ \hline \end{array} 0 & \begin{array}{c} 0 \\ 1 \end{array} \begin{array}{|c|c|} \hline & \\ \hline \end{array} 0 \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \end{array}$$

The partitions  $\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(6)}$  are illustrated from left to right as Young diagrams. In  $\nu^{(1)}$ , 0 and 1 on the left signify  $p_2^{(1)} = 0$  and  $p_1^{(1)} = 1$ . Looking on the right we see  $J_2^{(1)} = (0), J_1^{(1)} = (1, 1, 0, 0)$ . From (3.5) we have  $c(\nu) = -18$ , hence  $c(\nu, J) = -14$ .

**3.3. The bijection from RCs to paths.** We now describe the bijection  $\Phi : \text{RC}(\lambda, L) \rightarrow \mathcal{P}(\lambda, L)$ . Let  $(\nu, J) \in \text{RC}(\lambda, L)$ . We shall define a map  $\gamma : \text{RC}(\lambda, L) \rightarrow B$  which associates to  $(\nu, J)$  an element of  $B$ . Denote by  $\text{RC}_b(\lambda, L)$  the elements of  $\text{RC}(\lambda, L)$  such that  $\gamma(\nu, J) = b$ . We shall define a bijection  $\delta : \text{RC}_b(\lambda, L) \rightarrow \text{RC}(\lambda - \text{wt}(b), L - 1)$ . The disjoint union of these bijections then defines a bijection  $\delta : \text{RC}(\lambda, L) \rightarrow \bigsqcup_{b \in B} \text{RC}(\lambda - \text{wt}(b), L - 1)$ .

The bijection  $\Phi$  is defined recursively as follows. For  $b \in B$  let  $\mathcal{P}_b(\lambda, L)$  be the set of paths in  $B^{\otimes L}$  that have  $b$  as rightmost tensor factor. For  $L = 0$  the bijection  $\Phi$  sends the empty rigged configuration (the only element of the set  $\text{RC}(\lambda, L)$ ) to the empty path (the only element of  $\mathcal{P}(\lambda, L)$ ). Otherwise assume that  $\Phi$  has been defined for  $B^{\otimes(L-1)}$  and define it for  $B^{\otimes L}$  by the commutative diagram

$$(3.12) \quad \begin{array}{ccc} \text{RC}_b(\lambda, L) & \xrightarrow{\Phi} & \mathcal{P}_b(\lambda, L) \\ \delta \downarrow & & \downarrow \\ \text{RC}(\lambda - \text{wt}(b), L - 1) & \xrightarrow{\Phi} & \mathcal{P}(\lambda - \text{wt}(b), L - 1) \end{array}$$

where the right hand vertical map removes the rightmost tensor factor  $b$ . In short,

$$(3.13) \quad \Phi(\nu, J) = \Phi(\delta(\nu, J)) \otimes \gamma(\nu, J).$$

Here follows the main theorem of our paper.

**Theorem 3.2.**  $\Phi : \text{RC}(\lambda, L) \rightarrow \mathcal{P}(\lambda, L)$  is a bijection such that

$$(3.14) \quad c(\nu, J) = D(\Phi(\nu, J)) \quad \text{for all } (\nu, J) \in \text{RC}(\lambda, L).$$

#### 4. THE BIJECTION

In this section, for  $(\nu, J) \in \text{RC}(\lambda, L)$ , an algorithm is given which defines  $b = \gamma(\nu, J)$ , the new smaller rigged configuration  $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$  such that  $(\tilde{\nu}, \tilde{J}) \in \text{RC}(\rho, L - 1)$  where  $\rho = \lambda - \text{wt}(b)$ , and the new vacancy numbers in terms of the old.

Illustrating a rigged configuration as in Example 3.1 we call a row in  $\nu^{(a)}$  *singular* if its rigging (number on the right) is equal to the corresponding vacancy number  $p_i^{(a)}$ .

**4.1. Algorithm  $\delta$ .** Suppose you are at  $b = \Phi$  in the crystal graph  $B_0$  and set  $\ell_0 = 1$ . Repeat the following process for  $j = 1, 2, \dots$  until stopped. From  $b$  proceed by one step through an arrow of color  $a$ . Find the minimal integer  $i \geq \ell_{j-1}$  such that  $\nu^{(a)}$  has a singular row of length  $i$  and set  $\ell_j = i$ , reset  $b$  to be the sink of the arrow. If there is no such integer, then set  $\ell_j = \infty$  and stop. If there are two arrows sourcing from  $b$ , compare the minimal integers and take the smaller one. If the integers are the same, either one can be taken. The output of the algorithm does not depend on the choices by Lemma 2.1 (3).

We also use the notation  $\ell_k^{(a)} (= \ell_j)$  if at the  $j$ -th step the arrow has color  $a$  and it is the  $k$ -th one having color  $a$  from the beginning.

**4.2. New configuration.** The new configuration  $\tilde{\nu} = (\tilde{m}_i^{(a)})$  is changed to

$$(4.1) \quad \tilde{m}_i^{(a)} = m_i^{(a)} - \sum_{k=1}^{k_a} (\delta_{i, \ell_k^{(a)}} - \delta_{i, \ell_k^{(a)} - 1})$$

where  $k_a$  is the maximum of  $k$  such that  $\ell_k^{(a)}$  is finite.

**4.3. Change in vacancy numbers.** Let  $A$  be a statement, then  $\chi(A) = 1$  if  $A$  is true and  $\chi(A) = 0$  if  $A$  is false. Then from (3.10) one has

$$\tilde{p}_i^{(1)} - p_i^{(1)} = -1 + 2\chi(i \geq \ell_1^{(1)}) - \chi(i \geq \ell_1^{(2)}) - \chi(i \geq \ell_2^{(2)}) + 2\chi(i \geq \ell_2^{(1)}) - \chi(i \geq \ell_3^{(2)}).$$

Here we set  $\ell_k^{(a)} = \infty$  if  $k > k_a$ . This calculation is summarized in the following table.

•  $a = 1$

$[1, \ell_1^{(1)})$	$[\ell_1^{(1)}, \ell_1^{(2)})$	$[\ell_1^{(2)}, \ell_2^{(2)})$	$[\ell_2^{(2)}, \ell_2^{(1)})$	$[\ell_2^{(1)}, \ell_3^{(2)})$	$[\ell_3^{(2)}, \infty)$
-1	+1	0	-1	+1	0

The first row signifies the range of  $i$ , namely,  $[1, \ell_1^{(1)})$  means  $1 \leq i < \ell_1^{(1)}$  and the second row  $\tilde{p}_i^{(1)} - p_i^{(1)}$  in this range. Similarly one obtains the following tables for other  $a$ .

•  $a = 2$

$[1, \ell_1^{(1)})$	$[\ell_1^{(1)}, \ell_1^{(2)})$	$[\ell_1^{(2)}, \ell_1^{(3)})$	$[\ell_1^{(3)}, \ell_2^{(3)})$	$[\ell_2^{(3)}, \ell_2^{(2)})$	$[\ell_2^{(2)}, \min(\ell_2^{(1)}, \ell_3^{(3)})]$
0	-1	+1	0	-1	+1
$[\min, \max)$	$[\max, \ell_3^{(2)})$	$[\ell_3^{(2)}, \ell_4^{(3)})$	$[\ell_4^{(3)}, \infty)$		
0	-1	+1	0		

In this table min, max without  $(\cdot, \cdot)$  means the abbreviation of the previous parenthesis.

•  $a = 3$

$[1, \ell_1^{(2)})$	$[\ell_1^{(2)}, \ell_1^{(3)})$	$[\ell_1^{(3)}, \min(\ell_1^{(4)}, \ell_1^{(6)})]$	$[\min, \max)$	$[\max, \ell_2^{(3)})$
0	-1	+1	0	-1
$[\ell_2^{(3)}, \min(\ell_2^{(2)}, \ell_2^{(4)})]$		$[\min, \max)$	$[\max, \ell_3^{(3)})$	$[\ell_3^{(3)}, \min(\ell_2^{(6)}, \ell_3^{(2)})]$
+1		0	-1	+1
$[\min, \max)$	$[\max, \ell_4^{(3)})$	$[\ell_4^{(3)}, \ell_3^{(4)})$	$[\ell_3^{(4)}, \infty)$	
0	-1	+1	0	

•  $a = 4$

$[1, \ell_1^{(3)})$	$[\ell_1^{(3)}, \ell_1^{(4)})$	$[\ell_1^{(4)}, \min(\ell_1^{(5)}, \ell_2^{(3)})]$	$[\min, \max)$	$[\max, \ell_2^{(4)})$	$[\ell_2^{(4)}, \ell_3^{(3)})$
0	-1	+1	0	-1	+1
$[\ell_3^{(3)}, \ell_4^{(3)})$	$[\ell_4^{(3)}, \ell_3^{(4)})$	$[\ell_3^{(4)}, \ell_2^{(5)})$	$[\ell_2^{(5)}, \infty)$		
0	-1	+1	0		

•  $a = 5$

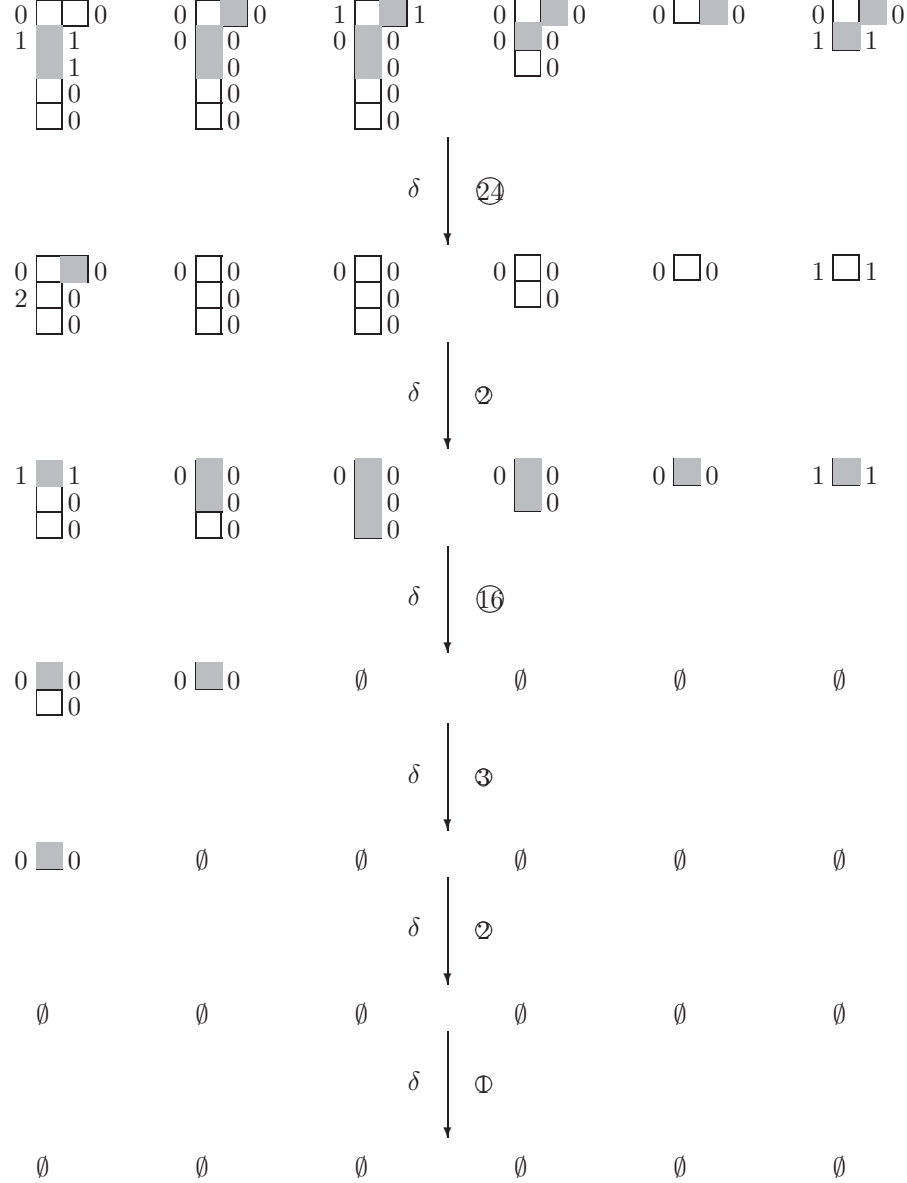
$[1, \ell_1^{(4)})$	$[\ell_1^{(4)}, \ell_1^{(5)})$	$[\ell_1^{(5)}, \ell_2^{(4)})$	$[\ell_2^{(4)}, \ell_3^{(4)})$	$[\ell_3^{(4)}, \ell_2^{(5)})$	$[\ell_2^{(5)}, \infty)$
0	-1	+1	0	-1	+1

•  $a = 6$

$[1, \ell_1^{(3)})$	$[\ell_1^{(3)}, \ell_1^{(6)})$	$[\ell_1^{(6)}, \ell_2^{(3)})$	$[\ell_2^{(3)}, \ell_3^{(3)})$	$[\ell_3^{(3)}, \ell_2^{(6)})$	$[\ell_2^{(6)}, \ell_4^{(3)})$	$[\ell_4^{(3)}, \infty)$
0	-1	+1	0	-1	+1	0



**Example 4.1.** The algorithm  $\Phi$  for the rigged configuration in Example 3.1 is described at each step  $\delta$  below.



Hence this rigged configuration corresponds to the path in Example 2.2 by  $\Phi$ .

**4.4. Inverse algorithm  $\tilde{\delta}$ .** For a given rigged configuration  $(\tilde{\nu}, \tilde{J})$  and  $b \in B$  the inverse algorithm  $\tilde{\delta}$  of  $\delta$  is described as follows. From  $b \in B$  go back the arrow in the crystal graph  $B_0$ . Let the maximal length of the singular row in  $\nu^{(a)}$  be  $\tilde{\ell}_0$ . Repeat the following process for  $j = 1, 2, \dots$  until we arrive at  $\Phi$ . Suppose the color of the arrow is  $a$ . Find the maximal integer  $i \leq \tilde{\ell}_{j-1}$  such that  $\nu^{(a)}$  has a singular row of length  $i$  and set  $\tilde{\ell}_j = i$ , reset  $b$  to be the source of the arrow. If there are two arrows ending at  $b$ , compare the maximal integers and take the larger one. If

the integers the same, either one can be taken. The output of the algorithm does not depend on the choices.

## 5. PROOF OF THEOREM 3.2

Theorem 3.2 is proved in this section. The following notation is used. Let  $(\nu, J) \in \text{RC}(\lambda, L)$ ,  $b = \gamma(\nu, J) \in B$ ,  $\rho = \lambda - \text{wt}(b)$ , and  $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$ . For  $(\nu, J) \in \text{RC}(\lambda, L)$ , define  $\Delta(c(\nu, J)) = c(\nu, J) - c(\delta(\nu, J))$ . The following lemma is essentially the same as [28, Lemma 5.1].

**Lemma 5.1.** *To prove that (3.14) holds, it suffices to show that it holds for  $L = 1$ , and that for  $L \geq 2$  with  $\Phi(\nu, J) = b_1 \otimes \cdots \otimes b_L$ , we have*

$$(5.1) \quad \Delta(c(\nu, J)) = -\alpha_1^{(1)},$$

and

$$(5.2) \quad H(b_{L-1} \otimes b_L) = \tilde{\alpha}_1^{(1)} - \alpha_1^{(1)}$$

where  $\alpha_1^{(1)}$  and  $\tilde{\alpha}_1^{(1)}$  are the lengths of the first columns in  $\nu^{(1)}$  and  $\tilde{\nu}^{(1)}$  respectively, and  $\delta(\nu, J) = (\tilde{\nu}, \tilde{J})$ .

There are five things that must be verified:

- (I)  $\rho$  is dominant.
- (II)  $(\tilde{\nu}, \tilde{J}) \in \text{RC}(\rho, L - 1)$ .
- (III)  $b$  can be appended to  $(\tilde{\nu}, \tilde{J})$  to give  $(\nu, J)$ .
- (IV) (5.1) in Lemma 5.1 holds.
- (V) (5.2) in Lemma 5.1 holds.

Parts (I) and (II) show that  $\delta$  is well-defined. Part (III) shows  $\delta$  has an inverse. Part (IV) and (V) suffice to prove that  $\Phi$  preserves statistics.

We need several preliminary lemmas on the convexity and nonnegativity of the vacancy numbers  $p_i^{(a)}$ .

**Lemma 5.2.** *For large  $i$ , we have*

$$p_i^{(a)} = \lambda_a$$

where  $\lambda_a$  is defined by  $\lambda = \sum_{a \in I_0} \lambda_a \bar{\Lambda}_a$ .

*Proof.* This follows from the formula for the vacancy number (3.3) and the constraint (3.1).  $\square$

Direct calculations show that

$$(5.3) \quad -p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} = L\delta_{a1}\delta_{i1} - 2m_i^{(a)} + \sum_{b \sim a} m_i^{(b)}.$$

In particular these equations imply the convexity condition

$$(5.4) \quad p_i^{(a)} \geq \frac{1}{2}(p_{i-1}^{(a)} + p_{i+1}^{(a)}) \quad \text{if } m_i^{(a)} = 0.$$

**Lemma 5.3.** *Let  $\nu$  be a configuration. The following are equivalent:*

- (1)  $p_i^{(a)} \geq 0$  for all  $i \in \mathbb{Z}_{>0}$ ,  $a \in I_0$ ;
- (2)  $p_i^{(a)} \geq 0$  for all  $i \in \mathbb{Z}_{>0}$ ,  $a \in I_0$  such that  $m_i^{(a)} > 0$ .

*Proof.* This follows immediately from Lemma 5.2 and the convexity condition (5.4).  $\square$

*Proof of (I).* Here we show  $\rho = \lambda - \text{wt}(b)$  is dominant. Suppose not. Let  $\lambda = \sum_{i \in I_0} \lambda_i \bar{\Lambda}_i$ . Since  $\varepsilon_i(b), \varphi_i(b) \leq 1$  for any  $i \in I_0$  and  $b \in B$ , in order to make  $\rho$  not dominant there exists  $a \in I_0$  such that  $\lambda_a = 0$  and  $\varphi_a(b) = 1$ . (There may be at most two such  $a$ , but the proof is uniform.) Let  $R$  be the route taken by the algorithm  $\delta$ . Although the arrow of color  $a$  sourcing from  $b$  is not taken by  $\delta$ , we include it into  $R$  as a terminal arrow from notational reason. Let  $(a_1, \dots, a_l)$  be colors of arrows in  $R$ . Let  $v_j$  be the source of the arrow of color  $a_j$ . Then  $a_l = a, v_l = b$ . Let  $\ell_j$  be the length of the singular row in  $\nu^{(a_j)}$  whose node is removed by  $\delta$ .

Let  $\ell$  be the largest part in  $\nu^{(a)}$ . We first show  $\ell > 0$ . Suppose  $\ell = 0$ . Then from (3.10) and Lemma 5.2 one gets

$$(5.5) \quad 0 = L\delta_{a1} + \sum_{c \sim a} Q_i^{(c)} \quad \text{for large } i.$$

However, this is a contradiction since along the route  $R$  there has to be some  $c$  such that  $c \sim a$  and a node in  $\nu^{(c)}$  was removed. There is only one exception:  $b = \Phi$  and  $a = 1$  case. This is also contradictory since the first term of the r.h.s. of (5.5) is positive. We can conclude  $\ell > 0$ .

The convexity condition (5.4) implies  $p_i^{(a)} = 0$  for all  $i \geq \ell$ . Equation (5.3) in turn yields  $m_i^{(c)} = 0$  for all  $i > \ell$  and  $c \sim a$ . Set  $k = \max\{1 \leq j < l \mid a_j \sim a\}$ . Then from Lemma 2.1 (4) there is an arrow of color  $a$  sourcing from  $v_j$  for any  $k < j < l$ , though by definition of  $a_k$  and  $a_l$ , all these arrows are not chosen by  $\delta$ . In view of the fact that  $m_i^{(a_k)} = 0$  for all  $i > \ell$  and  $\nu^{(a)}$  has a singular row of length  $\ell$ , one concludes that all length  $\ell$  rows of  $\nu^{(a)}$  had been removed before  $a_k$ . Thus we obtain

$$(5.6) \quad \#\{1 \leq j < l \mid a_j = a \text{ and } \ell_j = \ell\} = m_\ell^{(a)}.$$

Set  $i = \ell$  in (5.3). It yields

$$(5.7) \quad -p_{\ell-1}^{(a)} = L\delta_{a1}\delta_{\ell 1} - 2m_\ell^{(a)} + \sum_{c \sim a} m_\ell^{(c)}.$$

(5.6), Lemma 2.1 (1) imply  $\sum_{c \sim a} m_\ell^{(c)} \geq 2m_\ell^{(a)}$ , thus from (5.7) we deduce  $p_{\ell-1}^{(a)} = 0$  and  $\sum_{c \sim a} m_\ell^{(c)} = 2m_\ell^{(a)}$ . Let  $l_1 = \min\{1 \leq j < l \mid a_j = a \text{ and } \ell_j = \ell\}$ . Then the latter condition combined with Lemma 2.1 (1) and (5.6) imply that a node in each row of length  $\ell$  in  $\nu^{(c)}$  ( $c \sim a$ ) should be entirely removed during the process of the algorithm between  $j = l_1$  and  $j = l$ . Therefore length  $\ell$  rows of  $\nu^{(c)}$  ( $c \sim a$ ) are not removed between  $j = 1$  and  $j = l_1 - 1$ , which implies that  $\ell_j < \ell$  for all  $j \leq \max\{1 \leq j < l_1 \mid a_j \sim a\}$ . If  $m_{\ell-1}^{(a)} > 0$ , a node in all these rows should have been removed at the stage of  $j = l_1$  during the algorithm since these rows are singular and after  $j = l_1$  only length  $\ell$  rows are removed. Hence

$$(5.8) \quad \#\{1 \leq j < l \mid a_j = a \text{ and } \ell_j = \ell - 1\} = m_{\ell-1}^{(a)}.$$

This equality is valid also when  $m_{\ell-1}^{(a)} = 0$ .

Set  $i = \ell - 1$  in (5.3). It yields

$$(5.9) \quad -p_{\ell-2}^{(a)} = L\delta_{a1}\delta_{\ell-1,1} - 2m_{\ell-1}^{(a)} + \sum_{c \sim a} m_{\ell-1}^{(c)}.$$

(5.8), Lemma 2.1 (1) and (5.9) imply  $p_{\ell-2}^{(a)} = 0$  and  $\sum_{c \sim a} m_{\ell-1}^{(c)} = 2m_{\ell-1}^{(a)}$ . The latter condition implies  $\ell_j < \ell - 1$  for all  $j \leq \max\{1 \leq j < \ell_2 \mid a_j \sim a\}$  where  $\ell_2 = \min\{1 \leq j < \ell \mid a_j = a \text{ and } \ell_j \geq \ell - 1\}$ , since from Lemma 2.1 (1) a node in all the rows of length  $\ell - 1$  in  $\nu^{(c)}$  ( $c \sim a$ ) should be removed between  $j = \ell_2$  and  $j = \ell_1$ . We continue this procedure until  $j = 1$ , where

$$(5.10) \quad \sharp\{1 \leq j < l \mid a_j = a \text{ and } \ell_j = 1\} = m_1^{(a)},$$

$$(5.11) \quad -p_0^{(a)} = 0 = L\delta_{a1} - 2m_1^{(a)} + \sum_{c \sim a} m_1^{(c)}.$$

are established.

From (5.10), Lemma 2.1 (1) we have  $\sum_{c \sim a} m_1^{(c)} \geq 2m_1^{(a)}$ . It contradicts to (5.11) when  $a = 1$ . If  $a \neq 1$ , we have  $\sum_{c \sim a} m_1^{(c)} = 2m_1^{(a)}$ . This equation implies that a node in all the rows of length 1 in  $\nu^{(c)}$  ( $c \sim a$ ) should be removed during the process  $j \geq \min\{1 \leq j < l \mid a_j = a\}$ . However, it is a contradiction, since there exists a  $j$  such that  $a_j \sim a$  and  $j < \min\{1 \leq j < l \mid a_j = a\}$  by Lemma 2.1 (2). The proof is completed.  $\square$

*Proof of (II).* To prove the admissibility of  $(\tilde{\nu}, \tilde{J})$  we need to show

$$(5.12) \quad 0 \leq \tilde{J}_{\max}^{(a,i)} \leq \tilde{p}_i^{(a)}$$

for all  $i \geq 1, 1 \leq a \leq 6$  where  $\tilde{J}_{\max}^{(a,i)}$  stands for the largest part of  $\tilde{J}^{(a,i)}$ . In view of the definition of the algorithm  $\delta$  in §4.1 and the tables of  $\tilde{p}_i^{(a)} - p_i^{(a)}$  in §4.3, the condition (5.12) could only be violated when the following cases occur.

- (i) There exists a singular row of length  $i$  in  $\nu^{(a)}$  such that  $\ell_j \leq i < \ell_{j'}$  for some  $j < j'$ .
- (ii)  $m_{\ell_{j'}-1}^{(a)} = 0, p_{\ell_{j'}-1}^{(a)} = 0, \ell_j < \ell_{j'}$  for some  $j < j'$ .

In both cases  $\ell_{j'}$  corresponds to  $\nu^{(a)}$  and  $\ell_j$  to  $\nu^{(c)}$  such that  $c \sim a$  and  $j$  is the maximum that is less than  $j'$ .

We show (i) and (ii) cannot occur. Firstly, suppose (i) occurs. Then, by Lemma 2.1 (4), a node of this singular row of length  $i$  should have been removed by  $\delta$ , which is a contradiction. Suppose (ii) occurs. Let  $t$  be a maximal integer such that  $t < \ell_{j'}, m_t^{(a)} > 0$ ; if no such  $t$  exists set  $t = 0$ . By (5.4)  $p_{\ell_{j'}-1}^{(a)} = 0$  is only possible if  $p_i^{(a)} = 0$  for all  $t \leq i \leq \ell_{j'}$ . By (5.3) one finds that  $m_i^{(c)} = 0$  for all  $c \sim a, t < i < \ell_{j'}$ . Since  $\ell_j < \ell_{j'}$  this implies that  $\ell_j \leq t$ . If  $t = 0$ , it contradicts  $\ell_j \geq 1$ . Hence assume that  $t > 0$ . Since  $p_t^{(a)} = 0$  and  $m_t^{(a)} > 0$ , there is a singular row of length  $t$  in  $\nu^{(a)}$  and therefore  $\ell_{j'} = t$  by Lemma 2.1 (4), which contradicts  $t < \ell_{j'}$ .  $\square$

*Proof of (III).* Given  $(\tilde{\nu}, \tilde{J}) \in \mathcal{P}(\rho, L - 1)$  and  $b \in B$ , we want to show that one obtains the original  $(\nu, J) \in \mathcal{P}(\lambda, L)$  by the inverse procedure of  $\delta$ . However, once one notices from the tables in §4.3 that if a node is removed from a row of length  $\ell$  in  $\nu^{(a)}$ , then the difference  $\tilde{p}_i^{(a)} - p_i^{(a)} = +1$  for all  $\ell \leq i < \ell'$  where  $\ell'$  is the length of the singular row in  $\nu^{(c)}$  such that  $c \sim a$  removed by  $\delta$  after  $\ell$ , it is obvious that  $\tilde{\delta}$  gives the inverse procedure of  $\delta$ .  $\square$

*Proof of (IV).* Let  $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$ . Let  $\tilde{m}_i^{(a)}, \tilde{p}_i^{(a)}$  be for  $(\tilde{\nu}, \tilde{J})$ . Let  $\ell_k^{(a)}$  ( $1 \leq k \leq k_a$ ) be the length of the row a node of which is removed at the  $k$ -th time from  $\nu^{(a)}$

by the algorithm  $\delta$ . Then by (3.4),(3.7),(3.11) we have

$$(5.13) \quad \Delta(c(\nu, J)) = \frac{1}{2} \sum_{a,b} \sum_{j,k} C_{ab} \min(j, k) (m_j^{(a)} m_k^{(b)} - \tilde{m}_j^{(a)} \tilde{m}_k^{(b)}) \\ + \sum_j (L m_j^{(1)} - (L-1) \tilde{m}_j^{(1)}) + \sum_a \sum_{k=1}^{k_a} (p_{\ell_k^{(a)}}^{(a)} - \tilde{p}_{\ell_k^{(a)}-1}^{(a)}).$$

From (3.3) we obtain

$$p_{\ell_k^{(a)}}^{(a)} - \tilde{p}_{\ell_k^{(a)}-1}^{(a)} = \delta_{a1} (1 + (L-1) \delta_{\ell_k^{(a)}, 1}) \\ - \sum_{b,j} C_{ab} \left( \chi(j \geq \ell_k^{(a)}) m_j^{(b)} + \min(\ell_k^{(a)} - 1, j) \sum_{i=1}^{k_b} (\delta_{j, \ell_i^{(b)}} - \delta_{j, \ell_i^{(b)}-1}) \right).$$

Substituting (4.1) and the above into (5.13) one gets

$$\Delta(c(\nu, J)) = k_1 - \sum_j m_j^{(1)} - V,$$

where

$$V = \frac{1}{2} \sum_{a,b} \sum_{i=1}^{k_a} \sum_{j=1}^{k_b} C_{ab} (\delta_{\ell_i^{(a)} \ell_j^{(b)}} + \chi(\ell_i^{(a)} < \ell_j^{(b)})).$$

Use another notation for  $\ell_i^{(a)}$ . Namely, let  $\ell_j$  ( $j = 1, \dots, \ell$ ) be the successive length of the singular rows by  $\delta$ .  $V$  is calculated as

$$V = \frac{1}{2} \sum_{i,j=1}^{\ell} C_{a_i a_j} (\delta_{\ell_i \ell_j} + 2\chi(\ell_i < \ell_j)) \\ = \ell + \sum_{i < j} C_{a_i a_j} \\ = k_1.$$

Here we have used Lemma 2.1 (2) in the last equality. This completes the proof.  $\square$

*Proof of (V).* The proof is reduced to showing the following lemma.  $\square$

**Lemma 5.4.** *For  $(\nu, J) \in \text{RC}(\lambda, L)$  with  $L \geq 2$  set  $\gamma(\nu, J) = c, \gamma(\delta(\nu, J)) = b$ . Let  $\ell_k^{(a)}$  be the length of the singular row in  $\nu^{(a)}$  at the  $k$ -th time by the algorithm  $\delta$ . Define the following subsets of  $B \otimes B$ .*

$$S_1 = \{\mathfrak{1} \otimes \mathfrak{7} \mid j \geq 18\} \sqcup \{\mathfrak{2} \otimes \mathfrak{7} \mid j \geq 23\} \sqcup \{\mathfrak{3} \otimes \mathfrak{7} \mid j \geq 25\} \\ \sqcup \{\mathfrak{4} \otimes \mathfrak{7}, \mathfrak{7} \otimes \mathfrak{7} \mid j \geq 26\} \sqcup \{\mathfrak{7} \otimes \mathfrak{7} \mid i = 5, 8, 10, 13, 18\},$$

$$S_2 = \{\mathfrak{7} \otimes \mathfrak{7} \mid \mathfrak{7} \text{ can be reached by following some (possibly zero) arrows from } \mathfrak{7}\}.$$

Then we have

- (1)  $H(b \otimes c) = \begin{cases} -2 & \text{if } b \otimes c \in S_1 \\ 0 & \text{if } b \otimes c \in S_2 \\ -1 & \text{otherwise.} \end{cases}$
- (2)  $b \otimes c$  belongs to  $S_1$  if and only if  $\ell_1^{(1)} = \ell_2^{(1)} = 1$ .
- (3)  $b \otimes c$  belongs to  $S_2$  if and only if  $\ell_1^{(1)} > 1$ .

*Proof.* Checking (1) reduces to a finite calculation that can be confirmed by computer.

To prove (2) let  $\tilde{\ell}_k^{(a)}$  be the length of the row in  $\nu^{(a)}$  at the  $k$ -th time by the second  $\delta$ . We first show the condition  $\ell_1^{(1)} = \ell_2^{(1)} = 1$  is equivalent to

$$(5.14) \quad \ell_3^{(2)} \leq \tilde{\ell}_1^{(1)}, \ell_4^{(3)} \leq \tilde{\ell}_1^{(2)}, \ell_3^{(4)} \leq \tilde{\ell}_1^{(3)}, \ell_2^{(5)} \leq \tilde{\ell}_1^{(4)}, \tilde{\ell}_1^{(5)} = \infty.$$

Let  $R$  and  $\tilde{R}$  be the routes taken by the first and second algorithms  $\delta$ . Suppose  $\ell_1^{(1)} = \ell_2^{(1)} = 1$ . Then for all the arrows in  $R$  between the first one of color 1 and the second, the first  $\delta$  removes a node from a row of length 1, namely, removes the row. In view of the table for  $a = 1$  in §4.3 the length of the singular row after the first  $\delta$  should be no less than  $\ell_3^{(2)}$ . Hence we have  $\ell_3^{(2)} \leq \tilde{\ell}_1^{(1)}$ . For the next inequality view the table for  $a = 2$ . Since  $\ell_3^{(2)} \leq \tilde{\ell}_1^{(2)}$ , we get  $\ell_4^{(3)} \leq \tilde{\ell}_1^{(2)}$ . Proceeding similarly we obtain (5.14). Suppose (5.14) next and assume  $\ell_2^{(1)} > 1$ . Then after the first  $\delta$  there exists a singular row in  $\nu^{(1)}$  of length less than  $\ell_2^{(1)}$ , which means  $\tilde{\ell}_1^{(1)} < \ell_2^{(1)}$ . However, it contradicts to the first inequality of (5.14). Therefore, we have  $\ell_1^{(1)} = \ell_2^{(1)} = 1$ . The fact that (5.14) is equivalent to  $b \otimes c \in S_1$  is checked as follows. Suppose for instance that  $b = \mathcal{Q}$ . This means  $\tilde{\ell}_1^{(1)} < \infty$  and  $\tilde{\ell}_1^{(2)} = \infty$ . From the first inequality of (5.14) we have  $\ell_3^{(2)} < \infty$ , which implies  $c = \mathcal{D}$  for  $j \geq 23$ . Other cases can be checked similarly.

We are left to show (3). From the assumption  $\ell_1^{(1)} > 1$ , there are remaining singular rows after the first  $\delta$  which could be removed by the second  $\delta$ . Thus the “if” part is finished. To show the “only if” part, we assume  $\ell_1^{(1)} = 1$  and deduce a contradiction. From the table at §4.3,  $a = 1$ , the value for  $[\ell_1^{(1)}, \ell_1^{(2)}]$  is +1 while we have  $\ell_1^{(1)} = 1$ . Thus we have  $\tilde{\ell}_1^{(1)} \geq \ell_1^{(2)}$ . In view of the table at §4.3, at  $a = 2$ , the value for  $[\ell_1^{(2)}, \ell_1^{(3)}]$  is +1. Thus we find  $\tilde{\ell}_1^{(2)} \geq \ell_1^{(3)}$ . We can continue this procedure as follows. For  $\ell_k^{(a)} < \infty$  one can definitely find  $\tilde{\ell}_k^{(a)} < \infty$  by the assumption  $b \otimes c \in S_2$ . Imitating the way to show  $\tilde{\ell}_1^{(1)} \geq \ell_1^{(2)}$  and  $\tilde{\ell}_1^{(2)} \geq \ell_1^{(3)}$ , we can then find a pair  $(a', k')$  such that  $\ell_k^{(a)} \leq \ell_{k'}^{(a')}$  and  $\tilde{\ell}_k^{(a)} \geq \ell_{k'}^{(a')}$ . This procedure continues until we arrive at  $\ell_{k'}^{(a')} = \infty$ . However, the previous  $\tilde{\ell}_k^{(a)}$  should be finite since the second  $\delta$  can go further along the route taken by the first  $\delta$ . This contradicts to  $\tilde{\ell}_k^{(a)} \geq \ell_{k'}^{(a')} = \infty$ . The proof is finished.  $\square$

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